5.10 qualitative matrix stability

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Define: The sign function $\text{sign}(x) = \begin{cases} + , & \text{if } x > 0 \\ - , & \text{if } x < 0 \\ 0 , & \text{if } x = 0 \end{cases}$, $x \in \mathbb{R}$.

And
$$\text{sign}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \text{sign}(a_{11}) & \cdots & \text{sign}(a_{1n}) \\ \vdots & \ddots & \vdots \\ \text{sign}(a_{m1}) & \cdots & \text{sign}(a_{mn}) \end{pmatrix}$$

Ex. $\text{sign}\begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} + \\ + \\ - \\ 0 \end{pmatrix}$

$$-\text{sign}(x) = \text{sign}(-x). \implies \begin{cases} - + = - \\ - - = + \\ - 0 = 0. \end{cases}$$

Def. 5.8 A square matrix J whose sign pattern $\text{sign}(J) = Q$ is **qualitatively stable** if all matrices with the same sign pattern Q have eigenvalues with negative real parts.

Ex.

$$J = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \implies Q = \begin{bmatrix} - & + \\ 0 & - \end{bmatrix}$$

$\implies$ qualitatively stable

$\implies$ locally asymp. stable.

Note: Qualitatively stable $\implies$ locally asymp stable

(but not the converse)

Often easier than Routh Hurwitz or Gershgorin circle thm.

Define: A system of differential equations having the form
$$\frac{dx_i}{dt} = x_i\left(a_{i0} + \sum_{j=1}^{n} a_{ij} x_j\right), \quad i = 1, 2, \dots, n$$
is known as a **Lotka-Volterra system**, often used to study predator-prey interactions.

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Rewrite:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix}}_{\operatorname{diag}(X)} \left(\underbrace{\begin{bmatrix} a_{10} \\ a_{20} \\ \vdots \\ a_{n0} \end{bmatrix}}_{-b} + \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{\substack{A \\ \text{(interaction matrix)}}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{X} \right)$$

Equilibria: 0 is an equilibrium

$\bar{X}$ is an equilibrium if $A\bar{X} = b$

If $\det(A) \neq 0$, then $\bar{X} = A^{-1}b$ is a unique pos. eq.

If $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)^T > 0$, then (all $\bar{x}_i > 0$)

$$J(\bar{X}) = \operatorname{diag}(\bar{x})A = \begin{bmatrix} \bar{x}_1 a_{11} & \cdots & \bar{x}_1 a_{1n} \\ \vdots & \ddots & \vdots \\ \bar{x}_n a_{n1} & \cdots & \bar{x}_n a_{nn} \end{bmatrix}$$

$\Rightarrow Q = \operatorname{sign}(J) = \operatorname{sign}(A)$ for a Lotka-Volterra system.

Ex 5.23 Consider a 1-predator, 2-prey system

$$\frac{dx_1}{dt} = x_1(-a_{10} + a_{12}x_2 + a_{13}x_3)$$

$$\frac{dx_2}{dt} = x_2(a_{20} - a_{21}x_1)$$

$$\frac{dx_3}{dt} = x_3(a_{30} - a_{31}x_1 - x_3), \quad a_{ij} > 0.$$

Then a positive eq $\bar{X}$ must satisfy

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{21} & 0 & 0 \\ \cdots & & \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} a_{10} \\ -a_{20} \\ -a \cdots \end{bmatrix}$$

$$\begin{bmatrix} -a_{21} & 0 & 0 \\ -a_{31} & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} -a_{20} \\ -a_{30} \end{bmatrix}$$

Or $\quad \bar{x}_1 = \dfrac{a_{20}}{a_{21}} > 0, \quad \bar{x}_3 = a_{30} - a_{31}\bar{x}_1, \quad \bar{x}_2 = \dfrac{a_{10} - a_{13}\bar{x}_3}{a_{12}},$

So $\bar{X}$ is positive if $\bar{x}_2 > 0$ and $\bar{x}_3 > 0$.

Assume $\bar{X} > 0$.

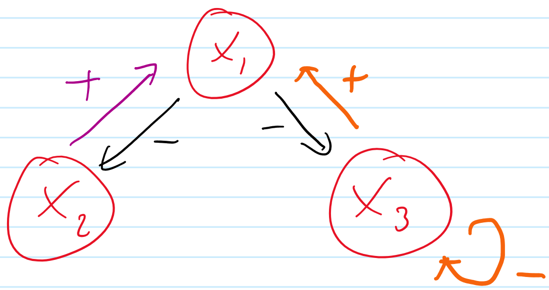
Then

$$Q = \text{sign}(\bar{J}) = \text{sign}(A) = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 0 & + & + \\ - & 0 & 0 \\ - & 0 & - \end{pmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}$$

Can write as signed digraph

Draw n nodes, and draw an edge $x_i \xrightarrow{+} x_j$ if $a_{ij} > 0$

$x_i \xrightarrow{-} x_j$ if $a_{ij} < 0$



Thm 5.12 (Necessary conditions for qualitative stability)

If $J \in \mathbb{R}^{n \times n}$ is qualitatively stable, and $Q = \text{sign}(\bar{J}) = (q_{ij})$, then

1. $q_{ii} \leq 0$ for all $i = 1, 2, \ldots, n$ (no positive feedback loops)
2. $q_{ii} < 0$ for some i (at least one negative feedback loop)
3. $\text{sign}(q_{ij}) = -\text{sign}(q_{ji})$ (opposite signs for interacting nodes)
4. Given a sequence $q_{i_0 i_1}, q_{i_1 i_2}, \ldots, q_{i_r i_0}$ (a path returning to the original node) containing $r+1 \geq 3$ distinct indices $i_0, \ldots, i_r$, at least one of the elements must be 0. (no cycles)

5. $\det(J) \neq 0$ (hyperbolic)

Note: we will often just assume this condition

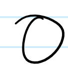
Def. 5.9 A **predation link** is a pair of nodes connected by edges going in opposite directions with opposite nonzero signs.

$$x_i \underset{-}{\overset{+}{\rightleftarrows}} x_j$$

A **predation community** is a maximal subgraph consisting of all connected predation links.

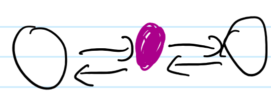
Color Test: Color each node with a negative feedback loop gray (purple), and all other nodes white. The color test is:

(1) There exists at least one white node.



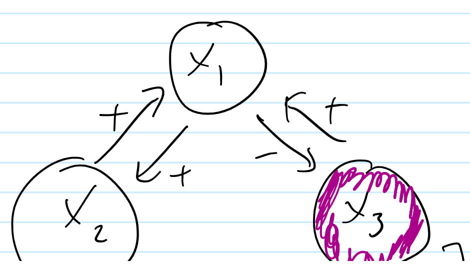
(2) Each white node is connected by a predation link to at least one other white node.

(3) Each gray (purple) node connected by a predation link to a white node is also connected by a predation link to at least one other white node.

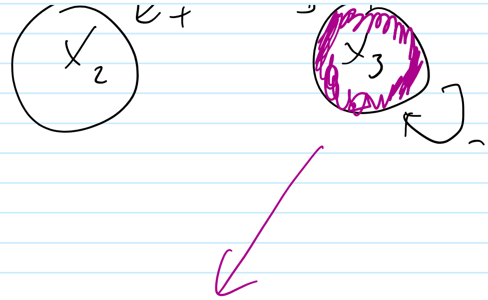


Thm 5.13 Let $Q = \text{sign}(J) = (a_{ij})$. If matrix J satisfies the five necessary conditions in Thm 5.12, and if in addition, each predation community fails at least one of the Color Test conditions, then J is qualitatively stable.

Ex.



1. No positive feedback loops
2. At least 1 neg. feedback loop (x_3)
3. All non self-loops come in opposite



3. All non self-loops come in opposite sign pairs.

4. No cycles

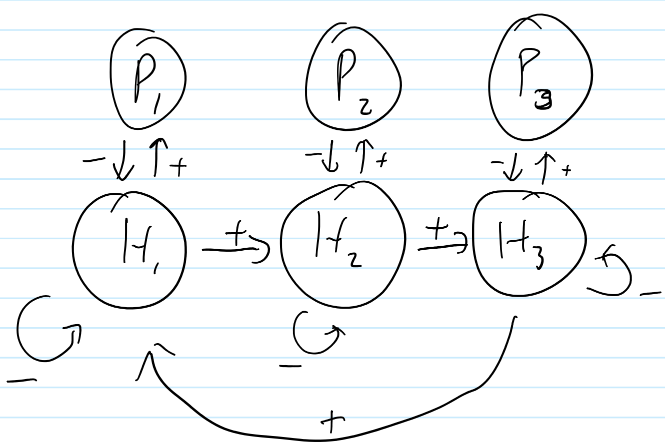
5. Assume $\det(J) \neq 0$. Often cannot tell from graph. } Assumption

Fails Color Test Cond 3: "gray" node only connected to 1 white node.

$\Rightarrow$ J is qualitatively stable if $\det(J) \neq 0$.

Ex.

$$
Q = \begin{array}{c c}
\begin{array}{cccccc} P_1 & P_2 & P_3 & H_1 & H_2 & H_3 \end{array} & \\
\begin{pmatrix}
0 & 0 & 0 & + & 0 & 0 \\
0 & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & 0 & 0 & + \\
- & 0 & 0 & - & 0 & + \\
0 & - & 0 & + & - & 0 \\
0 & 0 & - & 0 & + & -
\end{pmatrix} & \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ H_1 \\ H_2 \\ H_3 \end{array}
\end{array}
$$



Not qualitatively stable

because of closed loop. Also unpaired interaction arrows.